(-1)

Motivation & examples:

For appropriate $A \in GL_n(\mathbb{C})$,

$$\det(A) = \exp\left(\operatorname{Tr}\left(\operatorname{Log}(A)\right)\right)$$

Formula: (for $\operatorname{Tr}(\operatorname{Log}(A))$) $A = e^B$

$$\xi : [0,1] \longrightarrow GL_n(\mathbb{C})$$

$$\xi(\alpha) = e^{\alpha B} \qquad (\Rightarrow \xi(0) = 1,\ \xi(1) = e^B = A)$$

$$\det(A) = \exp\left(\int_0^1 \operatorname{Tr}\left(\xi'(\alpha)\, \xi(\alpha)^{-1}\right) d\alpha\right)$$

$$\int_0^1 \xi'(\alpha)\, \xi(\alpha)^{-1}\, d\alpha = B$$

De la Harpe–Skandalis determinant examples:

(1) $a \in \mathcal{A}$

$$2\pi i \Delta_\tau(e^a) = \tau(a) \mod \tau(K_0(\mathcal{A}))$$

$$\exp\left(\mathbb{C}/\mathbb{Z}\right) \qquad e^{r+is} = e^r e^{is}$$

$(0,\infty)$ infinite tube $\mathbb{Z}$ periodic

(2) $\tau : \mathcal{A} \to \mathbb{C}$

$$\tau(K_0(\mathcal{A})) = \mathbb{Z}$$

Then $\exp(i2\pi\Delta_\tau) : GL_\infty^0(\mathcal{A}) \longrightarrow \mathbb{C}^*$

is a group homomorphism

& $\exp(2\pi i \Delta_\tau)(e^a) = e^{\tau(a)} \quad \forall a \in \mathcal{A}$

$-0{,}5$

(3) Say that $\mathcal{A} = \mathbb{C}$

& $\tau : \mathcal{A} \to \mathbb{C}$ identity map.

Then $\exp(i2\pi\Delta_\tau)$ is the usual ~~identity~~ determinant on $GL_\infty(\mathbb{C})$

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(4) $(\mathcal{N},\tau)$ II_1 factor

$\tau(K_0(\mathcal{N})) = \mathbb{R}$

$\exp(\mathrm{Re}(i2\pi\Delta_\tau)) : GL_\infty(\mathcal{N}) \to \mathbb{R}^*_+$

is a surjective homomorphism of ~~of~~ groups.

& its restriction to $GL_1(\mathcal{N})$

is the Fuglede-Kadison determinant.

June 9th 2012

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de la Harpe — Skandalis Determinant

$1_a \in \mathcal{A}$ C^*-algebra

E Banach space

$\tau(ab) = \tau(ba)\ \forall a, b \in \mathcal{A}$

$\tau : \mathcal{A} \longrightarrow E$ tracial continuous linear ~~functional~~ map

$\xrightarrow{\text{induces}}$ $\tau : M_n(\mathcal{A}) \longrightarrow E$ tracial cont. linear map

$$[a_{j,k}] \longmapsto \sum_{j=1}^{n} \tau(a_{jj}) \qquad \forall n \geq 1$$

$GL_\infty(\mathcal{A}) = \bigcup_{n=1}^{\infty} GL_n(\mathcal{A})$ topological group

$GL^\circ_\infty(\mathcal{A})$ = connected component of identity

Next we define a map on the pathspace of $GL^\circ_\infty(\mathcal{A})$

$\xi : [\alpha_1, \alpha_2] \longrightarrow GL^\circ_\infty(\mathcal{A})$

piecewise continuously differentiable path.

Df: $$\tilde{\Delta}_\tau(\xi) =_{df} \frac{1}{2\pi i} \int_{\alpha_1}^{\alpha_2} \tau\left(\dot{\xi}(\alpha)\, \xi(\alpha)^{-1}\right) d\alpha$$

So $\tilde{\Delta}_\tau : \left\{ \begin{array}{l} \text{paths in} \\ GL^\circ_\infty(\mathcal{A}) \text{ as above} \end{array} \right\} \longrightarrow E$

$\tilde{\Delta}_\tau(\xi)$ depends only on the homotopy class of ξ.

Remark: $e \in M_n(\mathcal{A})$ projection

loop: $\xi_e : [0,1] \longrightarrow GL_n^\circ(\mathcal{A}) \subseteq GL_\infty^\circ(\mathcal{A})$

$$\alpha \longmapsto \exp(i 2\pi \alpha e) = e^{i2\pi\alpha} e + (1-e)$$

Th.: The map $e \longmapsto \xi_e$ extends to a group isomorphism

(Atiyah & Bott 1964)

$$K_0(\mathcal{A}) \xrightarrow{\cong} \pi_1\left(GL_\infty^\circ(\mathcal{A})\right)$$

Exercise: $\tilde{\Delta}_\tau(\xi_e) = \tau(e)$

Definition (de la Harpe – Skandalis 1984)

$$\Delta_\tau : GL_\infty^\circ(\mathcal{A}) \longrightarrow E / \tau(K_0(\mathcal{A}))$$

is given by: $x \in GL_\infty^\circ(\mathcal{A})$

ξ : **any** piecewise continuously differentiable path in $GL_\infty^\circ(\mathcal{A})$ with origin 1 & extremity x

Then $\Delta_\tau(x) =_{df} \tilde{\Delta}_\tau(\xi) \mod \tau(K_0(\mathcal{A}))$

Exercise: Δ_τ is well-defined & a group homomorphism.

Take

$$\tau = T : \mathcal{A} \to \mathcal{A} / [\mathcal{A},\mathcal{A}] \quad \text{quotient map}$$

$= E_u$ Notation

where $[\mathcal{A},\mathcal{A}] = \text{span}\{[a,b] =_{df} ab - ba \mid a, b \in \mathcal{A}\}$

"universal tracial continuous linear map"

$\mathcal{A}_{s.a} / [\mathcal{A},\mathcal{A}] \simeq \text{Aff}(T(\mathcal{A}))$ isometric order-preserving linear isomorph

Brown - Perera - Toms

Th (de la Harpe - Skandalis 1984)

$1_\mathcal{A} \in \mathcal{A}$ simple ∞-D AF-algebra

(i) $x \in GL^0(\mathcal{A})$

$\Delta_T(x) = 0 \iff \exists\, y_j, z_j \in GL^0(\mathcal{A}),\ 1 \le j \le 4,$

s.t. $x = \prod_{j=1}^{4} (y_j, z_j)$

$\boxed{(y_j, z_j) =_{df} y_j z_j y_j^{-1} z_j^{-1}}$

(ii) $u \in \mathcal{U}^0(\mathcal{A})$

$\Delta_T(u) = 0 \iff u$ is a finite product of multiplicative commutators in $\mathcal{U}^0(\mathcal{A})$

de la Harpe - Skandalis did NOT specify a # for this case & it is probably large

Th (Thomsen 1993)

$1_{\mathcal{a}} \in \mathcal{a}$ AH-algebra

s.t. each building block has form

$$M_{n_1}(C(X_1)) \oplus M_{n_2}(C(X_2)) \oplus \cdots \oplus M_{n_k}(C(X_k))$$

where $\begin{cases} X_j \text{ is compact connected } T_2 \text{ space} \\ \dim(X_j) \lesssim 2 \\ H^2(X_j, \mathbb{Z}) = \vec{0} \end{cases}$

$K_0(\mathcal{a})$ has large denominators

$x \in GL^\circ(\mathcal{a})$ (or $\mathcal{U}^\circ(\mathcal{a})$)

$\Delta_T(x) = 0 \iff x$ is a finite product of multiplicative commutators in $GL^\circ(\mathcal{a})$ ($\mathcal{U}^\circ(\mathcal{a})$ resp.)

Note: Thomsen does not give a bound for the # of commutators & no bound is implied by his proof.

So, potentially, the # of commutators can be arbitrarily large.

Df: (G, G_+) ordered group.

G has large denominators if

$\forall a \in G_+ \quad \forall k \in \mathbb{Z}^+$

$\exists\, b \in G_+ \ \&\ m \in \mathbb{Z}^+$

s.t. $kb \leq a \leq mb$

preliminary

Th I (N)

$1_{\mathcal{A}} \in \mathcal{A}$ separable simple ∞-D C^*-algebra

s.t.

either

(1) $\mathcal{A}$ has rr0, strict comparison & cancellation

or

(2) $\mathcal{A}$ is TAI

$x \in GL^\circ(\mathcal{A})$

Then $\Delta_T(x) = 0 \iff \exists\, y_j, z_j \in GL^\circ(\mathcal{A})$

s.t. $x = \prod_{j=1}^{10} (y_j, z_j)$

⌈For unitaries, we need 74 commutators.⌋

The #s are definitely preliminary

We will sketch part of proof for TAI case.

$GL_0(\mathcal{A})$

4 technical lemmas. (5)

1st 2 lemmas concerning approximating an element with $\Delta_\tau = 0$ by multiplicative commutators

(N) **Lemma A** $1_a \in \mathcal{A}$ separable simple TAI

$x \in GL^0(\mathcal{A})$

$\Delta_T(x) = 0$

Then $\forall \varepsilon > 0$, $\exists x_j, y_j \in GL^0(\mathcal{A})$ $(1 \le j \le 26)$

$\exists d \in \mathcal{A}$

s.t. $x = \left(\prod_{j=1}^{26} (x_j, y_j)\right) e^d$

$\|d\| < \varepsilon$

& $\tau(d) = 0$ $\forall \tau \in T(\mathcal{A})$

(N) **Lemma B** $1_a \in \mathcal{A}$ separable simple TAI

$x \in GL^0(\mathcal{A})$

$\|x - 1_a\| < \frac{1}{100}$

$\tau(\mathrm{Log}(x)) = 0$ $\forall \tau \in T(\mathcal{A})$

Then $\forall \varepsilon > 0$, $\exists x_j, y_j, z \in GL^0(\mathcal{A})$ $(1 \le j \le 12)$

s.t. $x = \left(\prod_{j=1}^{12} (x_j, y_j)\right) z$

$\|z - 1\| < \varepsilon$, $\left.\begin{matrix}\|x_j - 1\| \\ \|y_j - 1\|\end{matrix}\right\} < 16 \|x - 1\|^{1/2}$ & $\tau(\mathrm{Log}(z)) = 0$ $\forall \tau \in T(\mathcal{A})$

Next, we need some lemmas about "pushing invertibles into small projections modulo some commutators"

For both Lemmas C & D,

$\mathcal{A}$ is unital C^*-algebra

$p, q \in \mathrm{Proj}(\mathcal{A})$ s.t. $p \precsim q$ & $p + q = 1_{\mathcal{A}}$

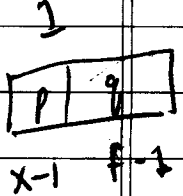
(dH-5) **Lemma C:** $x \in GL^0(\mathcal{A})$

$\|x - 1\| < \frac{1}{2}$ & $x - 1 \in p\mathcal{A}p$

Then $\exists\, y, z, f \in GL^0(\mathcal{A})$

s.t. $x = (y, z) f$, $f - 1 \in q\mathcal{A}q$,

$$\begin{cases} T(\mathrm{Log}(f)) = T(\mathrm{Log}(x)) \\ \|f - 1\| \leq \|x - 1\| \\ \|y - 1\| \leq 3\|x - 1\|^{1/2} \\ \|z - 1\| \leq 3\|x - 1\|^{1/2} \end{cases}$$



(dH-S) <u>Lemma D</u>: $x \in GL^{0}(\mathfrak{A})$

$$\|x - 1\| < \frac{1}{10}$$

Then $\exists\ y_1, z_1, y_2, z_2, e \in GL^{0}(\mathfrak{A})$ s.t.

$$x = (y_1, z_1)(y_2, z_2)\, e, \qquad e - 1 \in q\mathfrak{A}q$$

$$\tau(\operatorname{Log}(e)) = \tau(\operatorname{Log}(x))$$

$$\|e - 1\| \leq 4\,\|x - 1\|$$

$$\left.\begin{array}{l} \|y_j - 1\| \\ \|z_j - 1\| \end{array}\right\} \leq 5\,\|x - 1\|^{1/2} \qquad j = 1, 2$$

<u>Sketch of part of proof of Th I.</u>

$1 \in \mathfrak{A}$ simple ∞-D TAI $\qquad (GL^{0}(\mathfrak{A}))$

This argument is essentially the multiplicative version of Thierry Fack's argument for additive commutators in simple AF-algebras.

Step 1: $\exists$ sequences $\{p_n\}_{n=1}^{\infty}$, $\{q_n\}_{n=1}^{\infty}$, $\{r_n\}_{n=1}^{\infty}$ in $\mathrm{Proj}(\mathcal{A})$ s.t.

(i) $p_1 + q_1 + r_1 = 1_{\mathcal{A}}$

(ii) $p_n \overset{2}{\sim} q_n \overset{2}{\sim} r_n \quad \forall n \geq 1$

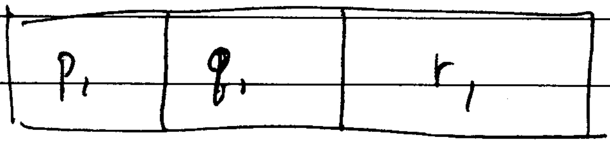
(iii) $r_m \perp r_n \quad \forall m \neq n$

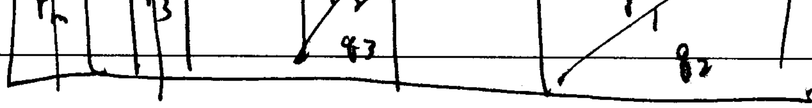
(iv) $r_n = p_{n+1} + q_{n+1} \quad \forall n \geq 1$

Follows from the weak divisibility of $\mathcal{A}$

This was proven for simple AF-algebras by Fack & used by de la Harpe & Skandalis

Thomsen generalized it to his class of algebras





Say $x \in GL^0(\mathfrak{a})$

$\Delta_T(x) = 0$

<u>Want</u>: x finite product of multiplicative commutators.

By Lemmas A & D,

WMA that

$x - 1 \in r_1 \mathfrak{a} r_1$, $T(\mathrm{Log}(x)) = 0$

$\|x - 1\| < \frac{1}{100}$

We construct, by induction, sequences

remainders, $\{x_n\}_{n=1}^{\infty}$, $\{y_n^j\}_{n=1}^{\infty}$, $\{z_n^j\}_{n=1}^{\infty}$ $(1 \le j \le 15)$

in $GL^0(\mathfrak{a})$ with $x_1 = x$

s.t.

$$\|x_n - 1_{\mathfrak{a}}\| < \frac{1}{10n^2}$$

$$x_n - 1_{\mathfrak{a}} \in r_n \mathfrak{a} r_n$$

$$T(\mathrm{Log}(x_n)) = 0 \qquad \forall n$$

$\Downarrow$

$$\left.\begin{array}{l}\|y_n^j - 1_{\mathcal{A}}\| \\ \|z_n^j - 1_{\mathcal{A}}\|\end{array}\right\} < \frac{16}{\sqrt{10}\,n} \qquad \left.\begin{array}{l}y_n^j - 1_{\mathcal{A}} \\ z_n^j - 1_{\mathcal{A}}\end{array}\right\} \in r_n \mathcal{A} r_n$$

$$1 \le j \le 14$$

$$\left.\begin{array}{l}y_n^{15} - 1_{\mathcal{A}} \\ z_n^{15} - 1_{\mathcal{A}}\end{array}\right\} \in (r_n + r_{n+1})\,\mathcal{A}\,(r_n + r_{n+1})$$

& $\forall n \ge 1$,

$$x_n = \left(\prod_{1 \le j \le 15} (y_n^j, z_n^j)\right) x_{n+1}$$

So
$$x = \left(\prod_{k=1}^{n}\left(\prod_{1 \le j \le 15}(y_n^j, z_n^j)\right)\right) x_{n+1}$$

$$\|x_{n+1} - 1\| \xrightarrow[n \to \infty]{} 0$$

The key is to turn this ∞ product of commutators into a finite product of commutators.

Up to this point, the proof is very much the multiplicative version of Fack's argument (for additive commutators). But since products (unlike addition) is noncommutative, we need to make a change

(11)

Replace $\left.\begin{matrix} y_n^j \\ z_n^j \end{matrix}\right\}$ with $\left.\begin{matrix} \tilde{y}_n^j \\ \tilde{z}_n^j \end{matrix}\right]$,

$\forall n \geq 1$,

$$x = \left[\left(\underbrace{\prod_{k=1}^{2n} \tilde{y}_k^1}_{\to \bar{y}^1}, \underbrace{\prod_{k=1}^{2n} \tilde{z}_k^1}_{\to \bar{z}^1} \right) \cdots \left(\underbrace{\prod_{k=1}^{2n} \tilde{y}_k^{14}}_{\to \bar{y}^{14}}, \underbrace{\prod_{k=1}^{2n} \tilde{z}_k^{14}}_{\to \bar{z}^{14}} \right) \right]$$

$$\left(\underbrace{\prod_{k=1}^{n} \tilde{y}^{15}_{2k-1}}_{\to \bar{y}^{15}}, \underbrace{\prod_{k=1}^{n} \tilde{z}^{15}_{2k-1}}_{\to \bar{z}^{15}} \right) \left(\underbrace{\prod_{k=1}^{n} y^{15}_{2k}}_{\to \bar{y}^{16}}, \underbrace{\prod_{k=1}^{n} z^{15}_{2k}}_{\to \bar{z}^{16}} \right) \underbrace{x_{n+1}}_{\to 1}$$

& all products are products of pairwise orthogonal factors which $\xrightarrow[k \to \infty]{} 1_a$

So all products converge in a

So $x = \prod_{j=1}^{15} (\bar{y}^j, \bar{z}^j)$ in $GL^0(a)$

$\square$